Exercise 1

Find the square roots of (a) $2i$; (b) $1 - \sqrt{3}i$ and express them in rectangular coordinates.

$$\textit{Ans.}\ (a)\quad \pm(1+i);\qquad (b)\quad \pm\frac{\sqrt{3}-i}{\sqrt{2}}.$$

Solution

For a nonzero complex number $z = re^{i(\Theta+2\pi k)}$, its square roots are

$$z^{1/2} = \left[re^{i(\Theta+2\pi k)}\right]^{1/2} = r^{1/2}\exp\left(i\frac{\Theta+2\pi k}{2}\right),\quad k = 0, 1.$$

Part (a)

The magnitude of $2i$ is $r = 2$, and the principal argument is $\Theta = \pi/2$.

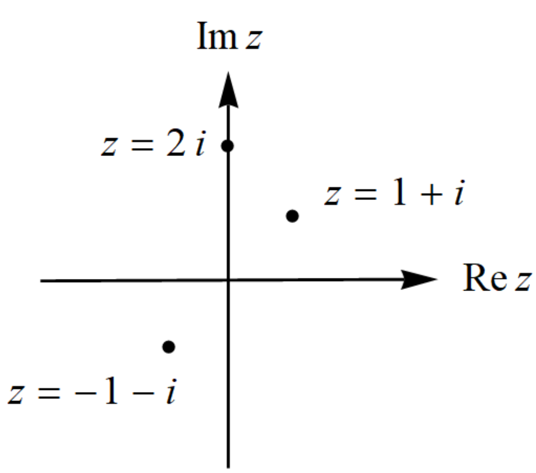
$$(2i)^{1/2} = 2^{1/2}\exp\left(i\frac{\frac{\pi}{2}+2\pi k}{2}\right),\quad k = 0, 1$$

The first root ($k = 0$) is

$$(2i)^{1/2} = 2^{1/2}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i,$$

and the second root ($k = 1$) is

$$(2i)^{1/2} = 2^{1/2}e^{i5\pi/4} = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -1 - i.$$



Part (b)

The magnitude and principal argument of $1 - \sqrt{3}i$ are respectively

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2 \quad \text{and} \quad \Theta = \tan^{-1} \frac{-\sqrt{3}}{1} = -\frac{\pi}{3},$$

so

$$(1 - \sqrt{3}i)^{1/2} = 2^{1/2} \exp\left(i\frac{-\frac{\pi}{3} + 2\pi k}{2}\right), \quad k = 0, 1.$$

The first root ($k = 0$) is

$$(1 - \sqrt{3}i)^{1/2} = 2^{1/2} e^{-i\pi/6} = \sqrt{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \frac{1}{\sqrt{2}}(\sqrt{3} - i),$$

and the second root ($k = 1$) is

$$(1 - \sqrt{3}i)^{1/2} = 2^{1/2} e^{i5\pi/6} = \sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -\frac{1}{\sqrt{2}}(\sqrt{3} - i).$$